

Geometric properties of Lagrangian mechanical systems

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Abstract

The geometry of a Lagrangian mechanical system is determined by its associated evolution semispray. We uniquely determine this semispray using the symplectic structure and the energy of the Lagrange space and the external force field. We study the variation of the energy and Lagrangian functions along the evolution and the horizontal curves and give conditions by which these variations vanish. We provide examples of mechanical systems which are dissipative and for which the evolution nonlinear connection is either metric or symplectic.

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Introduction

A geometric approach of Riemannian mechanical systems has been proposed recently by Munoz-Lecanda and Yaniz-Fernandez in [MY02]. Using techniques that are specific to Lagrange geometry, R.Miron, [Mir06I], introduced and investigated some geometric aspects of Finslerian and Lagrangian mechanical systems. In this work we extend such geometric investigation of Lagrangian mechanical systems. We determine the evolution semispray of a mechanical system by using the symplectic structure and the energy of the associated Lagrangian function and the external force field.

If the Lagrangian function is not homogeneous of second degree with respect to the velocity-coordinates, as it happens in the Riemannian and Finslerian framework, the energy of the system is different from the Lagrangian function and the evolution curves (solution of the Euler-Lagrange equations) are different from the horizontal curves of the system. In this paper we study the variation of both energy and Lagrangian function along the evolution curves and horizontal curves. As it has been shown for the Riemannian case, [MY02], we prove that

the energy is decreasing along the evolution curves of the system if and only if the external force field is dissipative.

The canonical nonlinear connection of a Lagrange manifold is the unique nonlinear connection that is metric and symplectic, as it has been shown in [Buc06]. Conditions by which the evolution nonlinear connection is either metric or symplectic are determined in terms of the symmetric or skew-symmetric part of a (1,1)-type tensor field associated with the external force field.

In the last part of the paper a special attention is paid to the particular case of Finslerian mechanical systems. Examples of dissipative mechanical systems are given.

1 Geometric structures on tangent bundle

In this section we introduce the geometric structures that live on the total space of tangent (cotangent) bundle, which we are going to use in this work such as: Liouville vector field, semispray, vertical and horizontal distribution.

For an n -dimensional C^∞ -manifold M , we denote by (TM, π, M) its tangent bundle and by (T^*M, τ, M) its cotangent bundle. The total space TM (T^*M) of the tangent (cotangent) bundle will be the phase space of the coordinate velocities (momenta) of our mechanical system. Let $(U, \phi = (x^i))$ be a local chart at some point $q \in M$ from a fixed atlas of C^∞ -class of the differentiable manifold M . We denote by $(\pi^{-1}(U), \Phi = (x^i, y^i))$ the induced local chart at $u \in \pi^{-1}(q) \subset TM$. The linear map $\pi_{*,u} : T_u TM \rightarrow T_{\pi(u)} M$ induced by the canonical submersion π is an epimorphism of linear spaces for each $u \in TM$. Therefore, its kernel determines a regular, n -dimensional, integrable distribution $V : u \in TM \mapsto V_u TM := \text{Ker} \pi_{*,u} \subset T_u TM$, which is called the *vertical distribution*. For every $u \in TM$, $\{\partial/\partial y^i|_u\}$ is a basis of $V_u TM$, where $\{\partial/\partial x^i|_u, \partial/\partial y^i|_u\}$ is the natural basis of $T_u TM$ induced by a local chart. Denote by $\mathcal{F}(TM)$ the ring of real-valued functions over TM and by $\mathcal{X}(TM)$ the $\mathcal{F}(TM)$ -module of vector fields on TM . We also consider $\mathcal{X}^v(TM)$ the $\mathcal{F}(TM)$ -module of vertical vector fields on TM . An important vertical vector field is $\mathbb{C} = y^i(\partial/\partial y^i)$, which is called the *Liouville vector field*.

The mapping $J : \mathcal{X}(TM) \rightarrow \mathcal{X}(TM)$ given by $J = (\partial/\partial y^i) \otimes dx^i$ is called the *tangent structure* and it has the following properties: $\text{Ker } J = \text{Im } J = \mathcal{X}^v(TM)$; $\text{rank } J = n$ and $J^2 = 0$. One can consider also the cotangent structure $J^* = dx^i \otimes (\partial/\partial y^i)$ with similar properties.

A vector field $S \in \mathcal{X}(TM)$ is called a *semispray*, or a second order vector field, if $JS = \mathbb{C}$. In local coordinates a semispray can be represented as follows:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}. \quad (1)$$

Integral curves of a semispray S are solutions of the following system of SODE:

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0. \quad (2)$$

A *nonlinear connection* N on TM is an n -dimensional distribution $N : u \in TM \mapsto N_u TM \subset T_u TM$ that is supplementary to the vertical distribution. This means that for every $u \in TM$ we have the direct sum

$$T_u TM = N_u TM \oplus V_u TM. \quad (3)$$

The distribution induced by a nonlinear connection is called the *horizontal distribution*. We denote by h and v the horizontal and the vertical projectors that correspond to the above decomposition and by $\mathcal{X}^h(TM)$ the $\mathcal{F}(TM)$ -module of horizontal vector fields on TM . For every $u = (x, y) \in TM$ we denote by $\delta/\delta x^i|_u = h(\partial/\partial x^i|_u)$. Then $\{\delta/\delta x^i|_u, \partial/\partial y^i|_u\}$ is a basis of $T_u TM$ adapted to the decomposition (3). With respect to the natural basis $\{\partial/\partial x^i|_u, \partial/\partial y^i|_u\}$ of $T_u TM$, we have the expression:

$$\frac{\delta}{\delta x^i} \Big|_u = \frac{\partial}{\partial x^i} \Big|_u - N_i^j(u) \frac{\partial}{\partial y^j} \Big|_u, \quad u \in TM. \quad (4)$$

The functions $N_j^i(x, y)$, defined on domains of induced local charts, are called the *local coefficients* of the nonlinear connection. The corresponding dual basis is $\{dx^i, \delta y^i = dy^i + N_j^i dx^j\}$.

It has been shown by M. Crampin [Cra71] and J. Grifone [Gri72] that every semispray determines a nonlinear connection. The horizontal projector h that corresponds to this nonlinear connection is given by:

$$h(X) = \frac{1}{2} (Id - \mathcal{L}_S J)(X) = \frac{1}{2} (X - [S, JX] - J[S, X]). \quad (5)$$

Local coefficients of the induced nonlinear connection are given by $N_j^i = \partial G^i / \partial y^j$.

2 Geometric structures on a Lagrange space

The presence of a regular Lagrangian on the tangent bundle TM determines the existence of some geometric structures one can associate to it such as: semispray, nonlinear connection and symplectic structure.

Consider $L^n = (M, L)$ a Lagrange space. This means that $L : TM \longrightarrow \mathbb{R}$ is differentiable of C^∞ -class on $\widetilde{TM} = TM \setminus \{0\}$ and only continuous on the null section. We also assume that L is a regular Lagrangian. In other words, the (0,2)-type, symmetric, d-tensor field with components

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} \text{ has rank } n \text{ on } \widetilde{TM}. \quad (6)$$

The Cartan 1-form θ_L of the Lagrange space can be defined as follows:

$$\theta_L = J^*(dL) = d_J L = \frac{\partial L}{\partial y^i} dx^i. \quad (7)$$

For a vector field $X = X^i(\partial/\partial x^i) + Y^i(\partial/\partial y^i)$ on TM , the following formulae are true:

$$\theta_L(X) = dL(JX) = d_J L(X) = (JX)(L) = \frac{\partial L}{\partial y^i} X^i. \quad (8)$$

The Cartan 2-form ω_L of the Lagrange space can be defined as follows:

$$\omega_L = d\theta_L = d(J^*(dL)) = dd_J L = d\left(\frac{\partial L}{\partial y^i} dx^i\right). \quad (9)$$

In local coordinates, the Cartan 2-form ω_L has the following expression:

$$\omega_L = 2g_{ij} dy^j \wedge dx^i + \frac{1}{2} \left(\frac{\partial^2 L}{\partial y^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial y^j} \right) dx^j \wedge dx^i. \quad (10)$$

We can see from expression (10) that the regularity of the Lagrangian L is equivalent with the fact that the Cartan 2-form ω_L has rank $2n$ on \widetilde{TM} and hence it is a symplectic structure on \widetilde{TM} . With respect to this symplectic structure, the vertical subbundle is a Lagrangian subbundle of the tangent bundle.

The canonical semispray of the Lagrange space L^n is the unique vector field \mathring{S} on TM that satisfies the equation

$$i_{\mathring{S}} \omega_L = -dE_L. \quad (11)$$

Here $E_L = \mathbb{C}(L) - L$ is the energy of the lagrange space L^n . The local coefficients \mathring{G}^i of the canonical semispray \mathring{S} are given by the following formula:

$$\mathring{G}^i = \frac{1}{4} g^{ik} \left(\frac{\partial^2 L}{\partial y^k \partial x^h} y^h - \frac{\partial L}{\partial x^k} \right). \quad (12)$$

Using the canonical semispray \mathring{S} we can associate to a regular Lagrangian L a canonical nonlinear connection with local coefficients given by expression $\mathring{N}_j^i = \partial \mathring{G}^i / \partial y^j$.

The horizontal subbundle NTM that corresponds to the canonical nonlinear connection is a Lagrangian subbundle of the tangent bundle TTM with respect to the symplectic structure ω_L . This means that $\omega_L(hX, hY) = 0$, $\forall X, Y \in \chi(TM)$. For a more detailed discussion regarding symplectic structures in Lagrange geometry we recommend [Ana03]. In local coordinates this implies the following expression for the symplectic structure ω_L :

$$\omega_L = 2g_{ij} \mathring{\delta} y^j \wedge dx^i. \quad (13)$$

The dynamical derivative that corresponds to the pair $(\mathring{S}, \mathring{N})$ is defined by $\nabla : \chi^v(TM) \longrightarrow \chi^v(TM)$ through:

$$\nabla \left(X^i \frac{\partial}{\partial y^i} \right) = \left(\mathring{S}(X^i) + X^j \mathring{N}_j^i \right) \frac{\partial}{\partial y^i}. \quad (14)$$

In terms of the natural basis of the vertical distribution we have

$$\nabla \left(\frac{\partial}{\partial y^i} \right) = \dot{N}_i^j \frac{\partial}{\partial y^j}. \quad (15)$$

Hence, \dot{N}_j^i are also local coefficients of the dynamical derivative. Dynamical derivative ∇ is the same with the covariant derivative D in [CMS96] or \mathcal{D}_Γ in [Kru97], where it is called the Γ -derivative. Dynamical derivative ∇ has the following properties:

- 1) $\nabla(X + Y) = \nabla X + \nabla Y, \forall X, Y \in \chi^v(TM),$
- 2) $\nabla(fX) = S(f)X + f\nabla X, \forall X \in \chi^v(TM), \forall f \in \mathcal{F}(TM).$

It is easy to extend the action of ∇ to the algebra of d-tensor fields by requiring for ∇ to preserve the tensor product. For the metric tensor g , its dynamical derivative is given by

$$(\nabla g)(X, Y) = S(g(X, Y)) - g(\nabla X, Y) - g(X, \nabla Y), \forall X, Y. \quad (16)$$

In local coordinates, we have:

$$g_{ij|} := (\nabla g) \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = \dot{S}(g_{ij}) - g_{im} \dot{N}_j^m - g_{mj} \dot{N}_i^m. \quad (17)$$

The canonical nonlinear connection \dot{N} is metric, which means that $\nabla g = 0$. In [Buc06] it is shown that the canonical nonlinear connection of a Lagrange space is the unique nonlinear connection that is metric and symplectic.

3 Geometric structures of Lagrangian mechanical systems

The dynamical system of a Lagrangian mechanical system is a semispray, which we call the evolution semispray of the system. Such semispray is uniquely determined by the symplectic structure and the energy of the underlying Lagrange space and the external force field. The energy of the system is decreasing if and only if the force field is dissipative.

The nonlinear connection we associate to the evolution semispray is called the evolution nonlinear connection. Conditions by which such nonlinear connection is either metric or symplectic are studied.

A *Lagrangian mechanical system* is a triple $\Sigma_L = (M, L, V)$, where (M, L) is a Lagrange space and $V = V^i(x, y)(\partial/\partial y^i)$ is a vertical vector field, which is called the external force field of the system. Using the metric tensor g_{ij} of the Lagrange space one can define the vertical one-form $\sigma = \sigma_i dx^i$, where $\sigma_i(x, y) = g_{ij}(x, y)V^j(x, y)$.

The external force field V is *dissipative* if $g(\mathbb{C}, V) = g_{ij}y^i V^j \leq 0$.

The evolution equations of the mechanical system Σ_L are given by the following Lagrange equations, [Mir06I]:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = \sigma_i, \quad y^i = \frac{dx^i}{dt}. \quad (18)$$

For a regular Lagrangian, Lagrange equations (18) are equivalent to the following system of second order differential equations:

$$\frac{d^2 x^i}{dt^2} + 2\hat{G}^i \left(x, \frac{dx}{dt} \right) = \frac{1}{2} V^i \left(x, \frac{dx}{dt} \right). \quad (19)$$

Here the functions \hat{G}^i are local coefficients of the canonical semispray of the Lagrange space, given by expression (12). Since $(1/2)V^i$ are components of a d-vector field on TM , from expression (19) we obtain that the functions

$$2G^i(x, y) = 2\hat{G}^i(x, y) - \frac{1}{2} V^i(x, y) \quad (20)$$

are coefficients of a semispray S , to which we refer to as the evolution semispray of the system. Therefore, the evolution semispray is given by $S = \hat{S} + (1/2)V$. Integral curves of the evolution semispray S are solutions of the SODE given by expression (19).

Theorem 3.1 *The evolution semispray S is the unique vector field on TM , solution of the equation*

$$i_S \omega_L = -dE_L + \sigma. \quad (21)$$

Proof. Since ω_L is a symplectic structure on TM , equation (21) uniquely determine a vector field S on TM . The vector field $V^i(\partial/\partial y^i)$ is the unique vector field that satisfies $i_{V^i(\partial/\partial y^i)} \omega_L = 2\sigma$. Using the linearity of equations (11) and (21) we can see that $S = \hat{S} + (1/2)V^i(\partial/\partial y^i)$ is the unique solution of equation(21). ■

Corollary 3.2 *The energy of the Lagrange space L^n is decreasing along the evolution curves of the mechanical system if and only if the external force field is dissipative.*

Proof. Using the fact that the evolution semispray S is solution of equation (21) and the skew-symmetry of ω_L we obtain $S(E_L) = dE_L(S) = \sigma(S) = \sigma_i y^i$. Along the evolution curves of the mechanical system, one can write this expression as follows:

$$\frac{d}{dt} (E_L) = \sigma_i \left(x, \frac{dx}{dt} \right) \frac{dx^i}{dt}. \quad (22)$$

Therefore, the energy is decreasing along the evolution curves if and only if $\sigma_i y^i \leq 0$. ■

Expression (22) has been obtained in [MY02] for the particular case of a Riemannian mechanical system. For the general case of Lagrangian mechanical system it has been obtained also in [Mir06I], using different techniques.

The evolution nonlinear connection of the mechanical system Σ_L has the local coefficients N_j^i given by

$$N_j^i = \frac{\partial G^i}{\partial y^j} = \frac{\partial \mathring{G}^i}{\partial y^j} - \frac{1}{4} \frac{\partial V^i}{\partial y^j} = \mathring{N}_j^i - \frac{1}{4} \frac{\partial V^i}{\partial y^j}. \quad (23)$$

Theorem 3.3 *The evolution nonlinear connection N is metric if and only if the $(0, 2)$ -type d-tensor field $\partial\sigma_i/\partial y^j$ is skew-symmetric.*

Proof. The evolution nonlinear connection is metric if and only if the dynamical covariant derivative of the metric tensor g_{ij} with respect to the pair (S, N) vanishes. This covariant derivative is given by

$$\begin{aligned} g_{ij|} &= S(g_{ij}) - g_{ik}N_j^k - g_{kj}N_i^k \\ &= \left(\mathring{S} + \frac{1}{2}V \right) (g_{ij}) - g_{ik} \left(\mathring{N}_j^k - \frac{1}{4} \frac{\partial V^k}{\partial y^j} \right) - g_{kj} \left(\mathring{N}_i^k - \frac{1}{4} \frac{\partial V^k}{\partial y^i} \right) \\ &= \mathring{S}(g_{ij}) - g_{ik}\mathring{N}_j^k - g_{kj}\mathring{N}_i^k + \frac{1}{4} \left(2V(g_{ij}) + g_{ik} \frac{\partial V^k}{\partial y^j} + g_{kj} \frac{\partial V^k}{\partial y^i} \right) \\ &= \frac{V^k}{4} \left(2 \frac{\partial g_{ij}}{\partial y^k} - \frac{\partial g_{ik}}{\partial y^j} - \frac{\partial g_{kj}}{\partial y^i} \right) + \frac{1}{4} \left(\frac{\partial \sigma_i}{\partial y^j} + \frac{\partial \sigma_j}{\partial y^i} \right) \\ &= \frac{1}{4} \left(\frac{\partial \sigma_i}{\partial y^j} + \frac{\partial \sigma_j}{\partial y^i} \right). \end{aligned} \quad (24)$$

In the above calculations we did use the fact that the canonical nonlinear connection \mathring{N} is metric and the Cartan tensor

$$C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4} \frac{\partial^3 L}{\partial y^i \partial y^j \partial y^k} \quad (25)$$

is totally symmetric.

The dynamical covariant derivative of the metric tensor g_{ij} with respect to the pair (S, N) is given by

$$g_{ij|} = \frac{1}{4} \left(\frac{\partial \sigma_i}{\partial y^j} + \frac{\partial \sigma_j}{\partial y^i} \right). \quad (26)$$

Consequently, the evolution nonlinear connection is metric if and only if the $(0, 2)$ -type d-tensor field $\partial\sigma_i/\partial y^j$ is skew-symmetric. ■

Theorem 3.4 *The evolution nonlinear connection is compatible with the symplectic structure if and only if the $(0, 2)$ -type d-tensor field $\partial\sigma_i/\partial y^j$ is symmetric.*

Proof. The evolution nonlinear connection N is compatible with the symplectic structure of the Lagrange space if and only if $\omega_L(hX, hY) = 0, \forall X, Y \in \chi(TM)$, where h is the corresponding horizontal projector.

Let us consider the almost symplectic structure:

$$\omega = 2g_{ij}\delta y^j \wedge dx^i, \quad (27)$$

with respect to which both horizontal and vertical subbundles are Lagrangian subbundles. Using expressions (13) and (23) the canonical symplectic structure can be expressed as follows:

$$\begin{aligned}
\omega_L &= 2g_{ij}\delta y^j \wedge dx^i = 2g_{ij} \left(\delta y^j + \frac{1}{4} \frac{\partial V^j}{\partial y^k} dx^k \right) \wedge dx^i \\
&= 2g_{ij}\delta y^j \wedge dx^i + \frac{1}{4} \left(\frac{\partial \sigma_i}{\partial y^j} - \frac{\partial \sigma_j}{\partial y^i} \right) dx^j \wedge dx^i \\
&= \omega + \frac{1}{2} F_{ij} dx^j \wedge dx^i.
\end{aligned} \tag{28}$$

Here F_{ij} is the helicoidal tensor of the mechanical system Σ_L introduced by R.Miron in [Mir06I]:

$$F_{ij} = \frac{1}{2} \left(\frac{\partial \sigma_i}{\partial y^j} - \frac{\partial \sigma_j}{\partial y^i} \right). \tag{29}$$

Therefore, the evolution nonlinear connection is compatible with the symplectic structure if and only if $\omega_L = \omega$ which is equivalent to the fact that the helicoidal tensor of the mechanical system Σ_L , given by expression (29), vanishes. ■

4 Variation of energy and Lagrangian functions

In this section we study the variation of energy and Lagrangian functions along the horizontal curves of the evolution nonlinear connection.

For a Lagrangian mechanical system Σ_L consider S the evolution semispray given by expression (20) and the evolution nonlinear connection given by expression (23). Let h be the corresponding horizontal distribution given by expression (5). Then hS is also a semispray, its integral curves are called horizontal curves of the evolution nonlinear connection. They are solutions of the following system of SODE:

$$\nabla \left(\frac{dx^i}{dt} \right) = \frac{d^2 x^i}{dt^2} + N_j^i \left(x, \frac{dx}{dt} \right) \frac{dx^j}{dt} = 0. \tag{30}$$

In the previous section we studied the variation of the energy function along the evolution curves of the system. We shall study now, the variation of the energy and Lagrangian functions along the horizontal curves (30). This way we can determine external force fields such that the Lagrangian or the energy functions are first integrals for the system (30).

Theorem 4.1 *Consider h the horizontal projector of the evolution nonlinear connection. We have the following formula for the horizontal differential operator d_h of the Lagrangian L :*

$$2d_h L = d_J(S(L)) - \sigma. \tag{31}$$

In local coordinates, formula (31) is equivalent with the following expression for the horizontal covariant derivative of the Lagrangian L :

$$2L|_i := 2 \frac{\delta L}{\delta x^i} = \frac{\partial}{\partial y^i}(S(L)) - \sigma_i. \quad (32)$$

Proof. We prove first the following formulae regarding the Cartan 1-form θ_L of the Lagrange space:

$$\mathcal{L}_S \theta_L = dL + \sigma. \quad (33)$$

By differentiating $\iota_S \theta_L = \mathbb{C}(L)$ we obtain $d\iota_S \theta_L = d\mathbb{C}(L)$. Using the expression of the Lie derivative $\mathcal{L}_S = d\iota_S + \iota_S d$ we obtain $\mathcal{L}_S \theta_L = \iota_S \theta_L + d\mathbb{C}(L)$. From the defining formulae (9) and (21) for ω_L and S we obtain $\mathcal{L}_S \theta_L = dE_L + \sigma + d\mathbb{C}(L) = dL + \sigma$

In order to prove (31) we have to show that for every $X \in \chi(TM)$, we have that

$$2(d_h L)(X) := 2dL(hX) = (JX)(S(L)) - \sigma(X).$$

Using formula (33) we obtain

$$\begin{aligned} 0 &= (\mathcal{L}_S \theta_L - dL - \sigma)(X) = S\theta_L(X) - \theta_L[S, X] - dL(X) - \sigma(X) \\ &= S((JX)(L)) - J[S, X](L) - dL(X) - \sigma(X) \\ &= [S, JX](L) + (JX)(S(L)) - J[S, X](L) - dL(X) - \sigma(X) \\ &= (JX)(S(L)) - dL(X - [S, JX] + J[S, X]) - \sigma(X) \\ &= (JX)(S(L)) - \sigma(X) - dL(2hX). \end{aligned}$$

Consequently, formula (31) is true.

Due to the linearity of the operators involved in formula (31) we have that formulae (31) and (32) are equivalent. ■

Corollary 4.2 *If the external force field of the mechanical system satisfies the equation:*

$$\frac{\partial V^k}{\partial y^i} y^i \frac{\partial L}{\partial y^k} = -2\mathbb{C}(\dot{S}(L)) \quad (34)$$

then the Lagrangian L is constant along the horizontal curves of the evolution nonlinear connection.

Proof. The Lagrangian L is constant along the horizontal curves of the evolution nonlinear connection if and only if $hS(L) = 0$. If we contract expression (32) by y^i we obtain

$$\begin{aligned} 2(hS)(L) &= 2L|_i y^i = 2 \frac{\delta L}{\delta x^i} y^i = \mathbb{C}(S(L)) - \sigma(S) \\ &= \mathbb{C}(\dot{S}(L)) + \frac{1}{2} \frac{\partial V^k}{\partial y^i} y^i \frac{\partial L}{\partial y^k}. \end{aligned} \quad (35)$$

Therefore we can see that $hS(L) = 0$ if and only if the external force field satisfies equation (34). ■

Theorem 4.3 *Consider h the horizontal projector of the evolution nonlinear connection. We have the following expression for the horizontal differential operator d_h of the energy E_L :*

$$d_h E_L(X) = -\omega_L(\mathring{S}, hX). \quad (36)$$

In local coordinates, this is equivalent with the following expression for the horizontal covariant derivative of the energy E_L :

$$E_{L|i} := \frac{\delta E_L}{\delta x^i} = 2g_{ij} \left(2\mathring{G}^j - \mathring{N}_k^j y^k \right) + \frac{1}{2} g_{jk} \frac{\partial V^j}{\partial y^i} y^k. \quad (37)$$

Proof. We have that $d_h E_L(X) = (dE_L)(hX) = -\omega_L(\mathring{S}, hX)$. Since h is the horizontal projector for the evolution nonlinear connection, for a vector field $X = X^i(\partial/\partial x^i) + Y^i(\partial/\partial y^i)$ on TM we have

$$hX = X^i \frac{\delta}{\delta x^i} = X^i \frac{\mathring{\delta}}{\delta x^i} + X^i \frac{1}{4} \frac{\partial V^j}{\partial y^i} \frac{\partial}{\partial y^j}.$$

Using expression (13) for the symplectic structure ω_L we have

$$\begin{aligned} d_h E_L(X) &= \left(-2g_{ij} \mathring{\delta} y^i \wedge dx^j \right) (\mathring{S}, hX) \\ &= 2g_{ij} X^i \left(2\mathring{G}^j - \mathring{N}_k^j y^k \right) + \frac{1}{2} g_{jk} \frac{\partial V^j}{\partial y^i} y^k X^i, \end{aligned} \quad (38)$$

and therefore we proved both formulae (36) and (37). ■

Corollary 4.4 *If the external force field of the mechanical system satisfies the equation:*

$$g_{jk} \frac{\partial V^j}{\partial y^i} y^k y^i = -4g_{ij} \left(2\mathring{G}^j - \mathring{N}_k^j y^k \right) y^i \quad (39)$$

then the energy E_L is constant along the horizontal curves of the evolution nonlinear connection.

5 Finslerian mechanical systems

Finsler geometry corresponds to the case when the Lagrangian function is second order homogeneous with respect to the velocity coordinates. This has various implications for the geometry of a Finsler space: the energy coincides with the fundamental function of the space and it is constant along the geodesic curves which are also horizontal curves for the canonical nonlinear connection. Therefore the geometry of a Finslerian mechanical system has some special features.

A Lagrange space $L^n = (M, L)$ reduces to a Finsler space $F^n = (M, F)$ if the Lagrangian function is second order homogeneous with respect to the

velocity coordinates. In this case we shall use the notation $F^2(x, y) = L(x, y)$ and therefore by using Euler's theorem for homogeneous functions we have:

$$\mathbb{C}(F^2) = \frac{\partial F^2}{\partial y^i} y^i = 2F^2.$$

A first consequence of the homogeneity condition is that the energy of a Finsler space coincides with the square of the fundamental function of the space: $E_{F^2} = \mathbb{C}(F^2) - F^2 = F^2$.

A Finslerian mechanical system is a triple $\Sigma_F = (M, F, V)$, where (M, F) is a Finsler space and $V = V^i(x, y)(\partial/\partial y^i)$ is a vertical vector field.

The evolution equations of the Finslerian mechanical system are given by Lagrange equations (18) where $L(x, y) = F^2(x, y)$, which are equivalent with the system of second order differential equations (19). The local coefficients \hat{G}^i of the canonical semispray \hat{S} of the Finsler space can be written in this case as follows:

$$2\hat{G}^i(x, y) = \gamma_{jk}^i(x, y)y^j y^k, \quad (40)$$

where $\gamma_{jk}^i(x, y)$ are Chrystoffel symbols of the metric tensor $g_{ij}(x, y)$. Therefore, the evolution semispray S of the mechanical system has the local coefficients $2G^i$ given by

$$2G^i(x, y) = \gamma_{jk}^i(x, y)y^j y^k - \frac{1}{2}V^i(x, y). \quad (41)$$

The evolution curves of the mechanical system are solutions of the following SODE:

$$\frac{d^2 x^i}{dt^2} + \gamma_{jk}^i \left(x, \frac{dx}{dt} \right) \frac{dx^j}{dt} \frac{dx^k}{dt} - \frac{1}{2}V^i \left(x, \frac{dx}{dt} \right) = 0. \quad (42)$$

Local coefficients N_j^i of the evolution nonlinear connection are given by expression (23). Due to the homogeneity conditions one can express them as follows:

$$N_j^i(x, y) = \gamma_{kj}^i(x, y)y^k - \frac{1}{4} \frac{\partial V^i}{\partial y^j}(x, y). \quad (43)$$

Therefore, the horizontal curves of the evolution nonlinear connection are solutions of the following SODE:

$$\frac{d^2 x^i}{dt^2} + \gamma_{jk}^i \left(x, \frac{dx}{dt} \right) \frac{dx^j}{dt} \frac{dx^k}{dt} - \frac{1}{4} \frac{\partial V^i}{\partial y^j} \left(x, \frac{dx}{dt} \right) \frac{dx^j}{dt} = 0. \quad (44)$$

Proposition 5.1 *Consider a Finslerian mechanical system $\Sigma_F = (M, F^2, V)$, with the external force field V homogeneous of order zero. Then the energy function $F^2(x, y)$ is constant along the horizontal curves of the evolution nonlinear connection.*

Proof. If the external force field V is homogeneous of order zero, then by Euler theorem we have $(\partial V^i / \partial y^j) y^j = 0$ and the horizontal curves of the evolution nonlinear connection given by expression (44) coincide with the geodesics of

the Finsler space. The energy function $F^2(x, y)$ is constant along the geodesic curves of the Finsler space.

One can obtain this result by using also expressions (39) or (34). The right hand side of both expression vanishes for a Finsler space, while the left hand side vanishes due to the homogeneity of the external force field. ■

For a Finsler space $(M, F^2(x, y))$, the local coefficients $2\mathring{G}^i(x, y)$ of the canonical semispray are given by expression (40) and therefore they are second order homogeneous with respect to the velocity variables. This implies that $2\mathring{G}^j = N_k^j y^k$ and equation (37) can be written as follows:

$$F_{|i}^2 = \frac{1}{2} g_{jk} \frac{\partial V^j}{\partial y^i} y^k = \frac{1}{2} \frac{\partial \sigma_k}{\partial y^i} y^k. \quad (45)$$

Here we did use the symmetry of the Cartan tensor (25) and the zeroth homogeneity of the metric tensor g_{ij} . If the helicoidal tensor of the mechanical system vanishes and the external force field is zero homogeneous then the horizontal covariant derivative of the energy function vanishes, in other words $F_{|i}^2 = 0$.

6 Examples

In this section we give examples of Finslerian and Lagrangian mechanical systems which have the properties that we studied in the previous sections.

1. Consider a Lagrangian mechanical system $\Sigma_L = (M, L, V)$, where $V = e\mathbb{C} = ey^i(\partial/\partial y^i)$ and e is a constant. We call this system a Liouville mechanical system. The evolution semispray S and nonlinear connection N have the local coefficients given by:

$$2G^i(x, y) = 2\mathring{G}^i - \frac{e}{2} y^i, \quad N_j^i = \mathring{N}_j^i - \frac{e}{4} \delta_j^i. \quad (46)$$

Therefore, the Liouville mechanical system $\Sigma_L = (M, L, e\mathbb{C})$ has some special properties.

Proposition 6.1 *The helicoidal tensor F_{ij} of the system vanishes. Consequently, the evolution nonlinear connection is compatible with the symplectic structure of the Lagrange space.*

Proof. The helicoidal tensor of the system is given by expression (29). Since $\sigma_i = eg_{ik}y^k$ we have the following expression:

$$\frac{\partial \sigma_i}{\partial y^j} = e \left(\frac{\partial g_{ik}}{\partial y^j} y^k + g_{ij} \right) = e (2C_{ijk} y^k + g_{ij}). \quad (47)$$

According to the above formula we have that the $(0, 2)$ -type d-tensor field $\partial \sigma_i / \partial y^j$ is symmetric and using Theorem 3.4, the evolution nonlinear connection is compatible with the symplectic structure of the Lagrange space. ■

Proposition 6.2 *The dynamical covariant derivative of the metric tensor g_{ij} with respect to the pair (S, N) is given by the following formula:*

$$g_{ij|} = \frac{e}{2} (2C_{ijk}y^k + g_{ij}). \quad (48)$$

Consequently, the evolution nonlinear connection is metric if and only if the metric tensor is homogeneous of order -1 .

Proof. Formula (48) follows immediately from expressions (47) and (26). We have that $g_{ij|} = 0$ if and only if

$$\frac{\partial g_{ij}}{\partial y^k} y^k = -1 \cdot g_{ij},$$

which is equivalent with the homogeneity of order -1 of the metric tensor g_{ij} . ■

If the Liouville mechanical system is also a Finslerian one, then $C_{ijk}y^k = 0$ and consequently $g_{ij|} = (e/2)g_{ij}$.

Proposition 6.3 *We assume the Liouville mechanical system is Finslerian. Then, the system is dissipative if and only if $e < 0$.*

Proof. The system is dissipative if and only if $0 > g(\mathbb{C}, V) = eg_{ij}y^i y^j = eF^2$, which holds true if and only if $e < 0$. ■

The above result holds true even if the Liouville mechanical system is truly Lagrangian, but in this case we have to ask for the supplementary condition $g_{ij}(x, y)y^i y^j > 0$. A sufficient condition for this is the positive definiteness of the metric tensor g_{ij} .

2. Consider the Finslerian mechanical system: $\Sigma_F = (M, F^2(x, y), (e/F)\mathbb{C})$. For this system, the external force field

$$V = \frac{ey^i}{F} \frac{\partial}{\partial y^i}$$

is zero homogeneous. According to the previous section the horizontal curves of the evolution nonlinear connection coincide with the geodesic curves of the Finsler space (M, F^2) .

Proposition 6.4 *For the Finslerian mechanical system Σ_F the following properties hold true.*

- i) *The helicoidal tensor F_{ij} of the system vanishes and hence the evolution nonlinear connection is compatible with the symplectic structure of the Finsler space.*
- ii) *The horizontal covariant derivative of the energy function vanishes. In other words $F^2_{|i} = 0$ and hence F^2 is constant along the horizontal curves of the evolution nonlinear connection.*

iii) The system is dissipative if and only if $e < 0$.

Proof.

i) Consider $y_i = g_{ij}y^j$. Using the symmetry of the Cartan tensor and the zero homogeneity of the metric tensor g_{ij} we obtain $\partial y_i / \partial y^j = g_{ij}$. Therefore we have:

$$\frac{\partial \sigma_i}{\partial y^j} = \frac{\partial}{\partial y^j} \left(\frac{e y_i}{F} \right) = \frac{e^2 g_{ij} - \sigma_i \sigma_j}{e F}. \quad (49)$$

From expression (49) we obtain that the helicoidal tensor F_{ij} vanishes.

ii) The horizontal covariant derivative of F^2 is given by expression (45). Using the symmetry of the tensor $\partial \sigma_k / \partial y^i$ and the zero homogeneity of the external force field we obtain

$$F^2_{|i} = \frac{1}{2} \frac{\partial \sigma_k}{\partial y^i} y^k = \frac{1}{2} \frac{\partial \sigma_i}{\partial y^k} y^k = 0. \quad (50)$$

iii) The system is dissipative if and only if $(e/F)g(\mathbb{C}, \mathbb{C}) < 0$, which is equivalent to $e < 0$. ■

An alternative approach of such systems can be obtained by means of Legendre transformations by using the theory of Cartan and Hamilton spaces. In this new framework, the Hamilton equations are used instead of Euler-Lagrange equations and the Hamiltonian vector field is used instead of the canonical semispray.

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